Remark: An important feature of Koszul formula is that we can compute Γ_{ij}^k directly from g_{ij} and its first derivatives, i.e.

$$\boxed{\Gamma_{ij}^k = F(g_{ij}, \partial_\ell g_{ij})}$$

Now, we can derive some sort of "Frenet-type" formulas by differentiating the moving basis $\{\partial_1, \partial_2, N\}$.

Prop: (Gauss) $D_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k + A_{ij} N$

(Weingarten) $D_{\partial_i} N = -g^{jk} A_{ij} \partial_k$ $\quad (*)$

Proof:

$$\begin{aligned} D_{\partial_i}\partial_j &= (D_{\partial_i}\partial_j)^T + (D_{\partial_i}\partial_j)^N \\ &= \nabla_{\partial_i}\partial_j + \langle D_{\partial_i}\partial_j, N\rangle N \\ &= \Gamma_{ij}^k \partial_k + A_{ij} N \end{aligned}$$

This proves the first equation. For the second equation, first notice that $\langle N, N\rangle \equiv 1$, therefore

$$0 = \partial_i \langle N, N\rangle = 2\langle D_{\partial_i} N, N\rangle$$

i.e. $D_{\partial_i} N$ is tangential!

On the other hand, since $\langle N, \partial_j \rangle \equiv 0$,

$$\langle D_{\partial_i} N, \partial_j \rangle = -\langle N, D_{\partial_i} \partial_j \rangle = -A_{ij}$$

So, if we let $D_{\partial_i} N = c_i^k \partial_k$, then

$$c_i^k g_{kj} = -A_{ij}$$

$$\Rightarrow \qquad c_i^k = -g^{kj} A_{ij}$$

This proves the second equation.

$\square$

Remark: From $(*)$ we see that the second f.f A_{ij} can be obtained from the "rate of change of the moving basis $\{\partial_1, \partial_2, N\}$", which in turn gives the Gauss and mean curvatures K and H.

§ Gauss and Codazzi equations

We are now ready to prove one of the most important theorems in classical differential geometry.

<u>Theorema Egregium</u>: (Gauss' Golden Theorem)

The Gauss curvature K is an intrinsic invariant, i.e. it depends only on the 1^{st} f.f. g_{ij} (and its higher derivatives)

The theorem above follows from a set of equations called "constraint equations". To understand these equations, we look at the following:

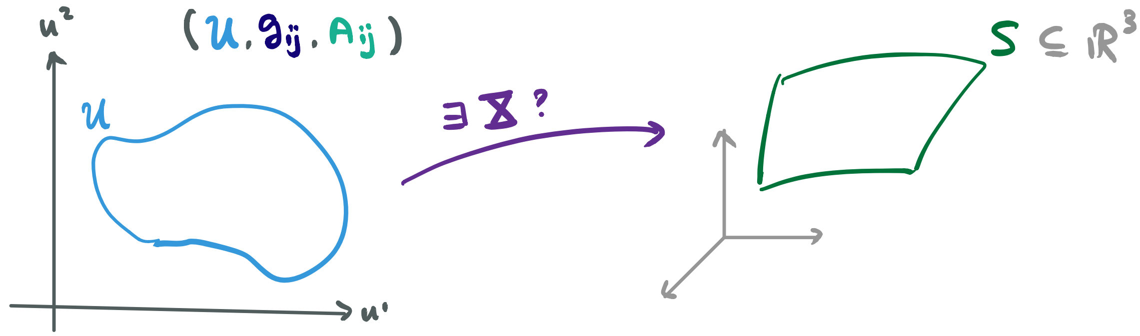
<u>Problem</u>: Given an open set $\mathcal{U} \subseteq \mathbb{R}^2$, and two smooth family of matrices defined on $\mathcal{U}$ s.t.

$$(g_{ij}) : \text{symmetric \& positive definite}$$

$$(A_{ij}) : \text{symmetric}$$

Can one find a parametrization $X : \mathcal{U} \to S \subseteq \mathbb{R}^3$ of a surface S s.t.

$$(g_{ij}) = 1^{st} \text{ f.f.} \quad \text{and} \quad (A_{ij}) = 2^{nd} \text{ f.f.} \quad ?$$



<u>Recall</u>: For curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, one can prescribe any curvature and torsion.

The situation is drastically different that the f.f.'s have to satisfy a natural set of "compatibility equations".

<u>Theorem</u>: If (g_{ij}) and (A_{ij}) are the 1^{st} & 2^{nd} f.f. of a surface in $\mathbb{R}^3$ under some local coordinate system, then we have:

Gauss equation:

$$\boxed{\partial_k \Gamma^{\ell}_{ij} - \partial_j \Gamma^{\ell}_{ik} + \Gamma^{p}_{ij}\Gamma^{\ell}_{pk} - \Gamma^{p}_{ik}\Gamma^{\ell}_{pj} = g^{\ell p}(A_{ij}A_{kp} - A_{ik}A_{jp})}$$

Codazzi equation:

$$\boxed{\partial_k A_{ij} - \partial_j A_{ik} + \Gamma^{p}_{ij} A_{pk} - \Gamma^{p}_{ik} A_{pj} = 0}$$

Proof: Let $\mathbf{X}(u^1,u^2): \mathcal{U} \to \mathbb{R}^3$ be a parametrization.

Recall the Gauss & Weingarten equations:

$$\begin{cases} \partial_i \partial_j \mathbf{X} = \Gamma_{ij}^k \partial_k + A_{ij} N \\ \partial_i N = -g^{jk} A_{ij} \partial_k \end{cases}$$

Since partial derivatives of any order commute:

$$\boxed{\partial_k(\partial_i \partial_j \mathbf{X}) = \partial_j(\partial_i \partial_k \mathbf{X})} \quad (\#)$$

We will show that Gauss & Codazzi equations follow from equating the tangential and normal components respectively in $(\#)$.

$$\partial_k(\partial_i \partial_j \mathbf{X}) \overset{\text{Gauss}}{=} \partial_k(\Gamma_{ij}^\ell \partial_\ell + A_{ij} N)$$

$$= (\partial_k \Gamma_{ij}^\ell)\partial_\ell + \Gamma_{ij}^\ell(\partial_k \partial_\ell \mathbf{X}) + (\partial_k A_{ij}) N + A_{ij}(\partial_k N)$$

$$\underset{\text{Weingarten}}{\overset{\text{Gauss}}{=}} (\partial_k \Gamma_{ij}^\ell)\partial_\ell + \Gamma_{ij}^\ell(\Gamma_{k\ell}^p \partial_p + A_{k\ell} N) + (\partial_k A_{ij}) N + A_{ij}(-g^{p\ell} A_{kp} \partial_\ell)$$

(grouping tangential and normal terms and renaming some dummy indices)

$$\partial_k(\partial_i \partial_j \mathbf{X}) = \left(\partial_k \Gamma^{\ell}_{ij} + \Gamma^{p}_{ij}\Gamma^{\ell}_{kp} - g^{p\ell} A_{ij} A_{kp} \right) \partial_\ell + \left(\partial_k A_{ij} + \Gamma^{\ell}_{ij} A_{k\ell} \right) N$$

Switching the indices k & j, we obtain

$$\partial_j(\partial_i \partial_k \mathbf{X}) = \left(\partial_j \Gamma^{\ell}_{ik} + \Gamma^{p}_{ik}\Gamma^{\ell}_{jp} - g^{p\ell} A_{ik} A_{jp} \right) \partial_\ell + \left(\partial_j A_{ik} + \Gamma^{\ell}_{ik} A_{j\ell} \right) N$$

Equating the tangential part $\Rightarrow$ Gauss equation.

Equating the normal part $\Rightarrow$ Codazzi equation.

$\square$

It turns out that these are all the equations constraining (g_{ij}) and (A_{ij}).

<u>Bonnet's Theorem</u>: Given (g_{ij}) & (A_{ij}) on an open set $\mathcal{U}$ of $\mathbb{R}^2$ satisfying Gauss & Codazzi equations, then

$$\exists \text{ parametrization } \mathbf{X} : \mathcal{U} \to \mathbb{R}^3$$

with $(g_{ij}) = 1^{st}$ f.f. & $(A_{ij}) = 2^{nd}$ f.f.

<u>Proof</u>: Omitted.

Proof of Theorema Egregium: (i.e. K is intrinsic.)

Recall: $$\boxed{K = \frac{\det(A_{ij})}{\det(g_{ij})}}$$

It suffices to show that $\det(A_{ij}) = A_{11}A_{22} - A_{12}^2$ is intrinsic (ie. depends only on g_{ij} and its derivatives)

Take $i=j=1$, $k=l=2$ in Gauss equation:

$$\partial_2 \Gamma_{11}^2 - \partial_1 \Gamma_{12}^2 + \Gamma_{11}^p \Gamma_{p2}^2 - \Gamma_{12}^p \Gamma_{p1}^2 = g^{2p}(A_{11}A_{2p} - A_{12}A_{1p})$$

$$= g^{21}\underbrace{(A_{11}A_{21} - A_{12}A_{11})}_{=0} + g^{22}(A_{11}A_{22} - A_{12}A_{12})$$

$$\Rightarrow \det(A_{ij}) = \underbrace{\frac{1}{g^{22}}\left(\partial_2 \Gamma_{11}^2 - \partial_1 \Gamma_{12}^2 + \Gamma_{11}^p \Gamma_{p2}^2 - \Gamma_{12}^p \Gamma_{p1}^2\right)}_{\text{depends only on } g_{ij} \text{ (up to } 2^{\text{nd}} \text{ derivatives).}}$$

$\square$